2. Sensenig, C. B., Instability of thick elastic solids. Commun. Pure and Appl. Math., Vol. 17, №4, 1964.
3. Southwell, R. V., On the general theory of elastic stability. Phil. Trans. Roy. Soc., London, Ser. A, Vol. 213, pp. 187-244, 1913.

Translated by M. D. F.

ASYMPTOTIC ANALYSIS OF SOME PLANE PROBLEMS OF THE THEORY OF ELASTICITY WITH COUPLE STRESSES

PMM Vol. 32, №6, 1968, pp. 1070-1074
R. M. BERGMAN
(Moscow)
(Received May 16, 1968)

The problem of stress concentration around curved holes without angular points, and the problem of an oscillating loading applied to the boundary of a half-plane are considered. The purpose is to investigate those properties of their solutions which result form smallness of the parameter l. The system is reduced to one equation, and an asymptotic method in the version of Vishik and Liusternik [1] is applied to solve it. For the concentration problem it is shown that if the solution by customary theory is known, then the solution by couple-stress theory can easily be constructed in a first approximation, and that couple-stress theory yields only an insignificant refinement. In the half-plane problem it is shown that the correction to the corresponding classical problem will be essential only in the case of rapid oscillation of the boundary conditions, i. e. when the state of stress being studied is of edge character. Another version of the asymptotic analysis of nonclassical problems of elasticity theory is given for fibrous media in [2].

1. In a Cartesian coordinate system the plane strain relationship in couple-stress elasticity theory, as presented in [3], are:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0,\quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0,\quad \frac{\partial \mu_x}{\partial x}+\frac{\partial \mu_y}{\partial y}+\tau_{xy}-\tau_{yx}=0 \qquad (1.1)$$

$$\varepsilon_x=\frac{1+\nu}{E}\left[\sigma_x-\nu(\sigma_x+\sigma_y)\right],\qquad \varepsilon_y=\frac{1+\nu}{E}\left[\sigma_y-\nu(\sigma_x+\sigma_y)\right]$$

$$\varepsilon_{xy}=\frac{1+\nu}{2E}(\tau_{xy}+\tau_{yx}),\quad \chi_x=\frac{1}{4Gl^2}\mu_x,\quad \chi_y=\frac{1}{4Gl^2}\mu_y$$

$$\varepsilon_x=\frac{\partial u}{\partial x},\qquad \varepsilon_y=\frac{\partial v}{\partial y},\qquad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\chi_x=\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\,\partial y}\right),\qquad \chi_y=\frac{1}{2}\left(\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial^2 u}{\partial y^2}\right)$$

Here $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$ and μ_x, μ_y are components of the force and moment stress tensors $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ and χ_x, χ_y are components of the strain and bending-torsion tensors; u, v the components of the displacement vector; E the Young's modulus, G the shear modulus; ν the Poisson coefficient; l the characteristic length of the material which we shall henceforth consider small as compared with the minimum radius of curvature of the hole.

Utilizing (1. 1), we obtain two equations in the displacements

$$\left[\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-l^2\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)\right]u+\left[\frac{1}{1-2\nu}\frac{\partial^2}{\partial x\,\partial y}+l^2\left(\frac{\partial^4}{\partial x^3\partial y}+\frac{\partial^4}{\partial y^3\partial x}\right)\right]v=0 \tag{1.2}$$

$$\left[\frac{1}{1-2\nu}\frac{\partial^2}{\partial x\,\partial y}+l^2\left(\frac{\partial^4}{\partial x^3\partial y}+\frac{\partial^4}{\partial y^3\partial x}\right)\right]u+\left[\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}-l^2\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^2\partial x^2}\right)\right]v=0$$

Let us introduce the potential function F by using Formulas

$$u=\left[\frac{2(1-\nu)}{1-2\nu}\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}-l^2\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^2\partial x^2}\right)\right]F$$

$$v=-\left[\frac{1}{1-2\nu}\frac{\partial^2}{\partial x\,\partial y}+l^2\left(\frac{\partial^4}{\partial x^3\partial y}+\frac{\partial^4}{\partial y^3\partial x}\right)\right]F$$

Then the second equation in (1.2) is satisfied identically, and the first will yield the governing equation for the function F

$$\nabla^2\nabla^2F-l^2\nabla^2\nabla^2\nabla^2F=0 \tag{1.3}$$

Again returning to (1.1), we express all the stresses in terms of F

$$\sigma_x=2G\left[\frac{2-\nu}{1-2\nu}\frac{\partial^3}{\partial x\,\partial y^2}+\frac{1-\nu}{1-2\nu}\frac{\partial^3}{\partial x^3}-l^2\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\nu}{1-2\nu}\nabla^2\right)\nabla^2\right]F$$

$$\sigma_y=-2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial^3}{\partial x\,\partial y^2}-\frac{\nu}{1-2\nu}\frac{\partial^3}{\partial x^3}+l^2\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2}+\frac{\nu}{1-2\nu}\nabla^2\right)\nabla^2\right]F$$

$$\tau_{xy}=2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial^3}{\partial y^3}-\frac{\nu}{1-2\nu}\frac{\partial^3}{\partial x^2\partial y}-l^2\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}-\frac{1-\nu}{1-2\nu}\nabla^2\right)\nabla^2\right]F$$

$$\tau_{yx}=2G\left[\frac{1-\nu}{1-2\nu}\frac{\partial^3}{\partial y^3}-\frac{\nu}{1-2\nu}\frac{\partial^3}{\partial x^2\partial y}-l^2\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}+\frac{1-\nu}{1-2\nu}\nabla^2\right)\nabla^2\right]F$$

$$\mu_x=-\frac{4(1-\nu)Gl^2}{1-2\nu}\frac{\partial^2}{\partial x\,\partial y}\nabla^2F,\qquad \mu_y=-\frac{4(1-\nu)Gl^2}{1-2\nu}\frac{\partial^2}{\partial y^2}\nabla^2F \tag{1.4}$$

Therefore, the system (1.1) is reduced to one by the introduction of the potential function F. Such a method may result in the loss of some solutions as, for example, has been shown in [4]. This circumstance indeed causes certain difficulties in the problem under consideration. But since they are successfully avoided, and the solution of couple-stress elasticity theory problems is unique [5], this will not affect the final result. We call A the representation of the stresses in terms of the function F by means of (1.4). It is nonsymmetric in x, y, and hence, is not unique. Replacing the minus by a plus sign in the last two expressions in (1.4), and interchanging x and y in each of the relationships in (1.4), we obtain another representation of the stresses in terms of some potential function which we denote by Φ. Let B designate this representation. The function Φ should also be a solution of (1.3) since it is symmetric in x, y.

Let us transform to curvilinear coordinates α, β by putting

$$x=x(\alpha,\beta),\qquad y=y(\alpha,\beta)$$

and considering x, y to be sufficiently smooth functions of α, β. Without limiting the generality, α, β can be considered an isothermal network, i.e. we assume that

$$\frac{\partial x}{\partial\alpha}=\frac{\partial y}{\partial\beta},\qquad \frac{\partial y}{\partial\alpha}=-\frac{\partial x}{\partial\beta} \tag{1.5}$$

Let us write the stress tensor components for A in these coordinates by using the general transformation formulas for contravariant tensor components

$$\sigma_\alpha = 2G\left[a_1 \frac{\partial^3}{\partial\beta^3} + \ldots + l^2\left(a_2 \frac{\partial^5}{\partial\beta^5} + \ldots\right)\right] F$$

$$\sigma_\beta = 2G\left[b_1 \frac{\partial^3}{\partial\beta^3} + \ldots + l^2\left(b_2 \frac{\partial^5}{\partial\beta^5} + \ldots\right)\right] F$$

$$\tau_{\alpha\beta} = 2G\left[c_1 \frac{\partial^3}{\partial\beta^3} + \ldots + l^2\left(c_2 \frac{\partial^5}{\partial\beta^5} + \ldots\right)\right] \cdot F$$

$$\tau_{\beta\alpha} = 2G\left[c_1 \frac{\partial^3}{\partial\beta^3} + \ldots + l^2\left(d_2 \frac{\partial^5}{\partial\beta^5} + \ldots\right)\right] F$$

$$\mu_\alpha = -\frac{4(1-\nu)Gl^2}{1-2\nu} p_1 \frac{\partial^4 F}{\partial\beta^4} + \ldots \qquad \mu_\beta = -\frac{4(1-\nu)Gl^2}{1-2\nu} q_1 \frac{\partial^4 F}{\partial\beta^4} + \ldots \tag{1.6}$$

Only higher derivatives with respect to β have been written down in (1.6). The coefficients therein have been expressed in terms of partial derivatives of x and y with respect to α and β. Let us introduce the notation

$$h = \left[\left(\frac{\partial x}{\partial\beta}\right)^2 + \left(\frac{\partial y}{\partial\beta}\right)^2\right]^{1/2}, \qquad h_0 = h\,|_{\beta=0}$$

Taking (1.5) into account we obtain

$$q_1 = \frac{\partial y}{\partial\beta}\frac{1}{h^6} \tag{1.7}$$

Passing to α, β in the representation B, we obtain formulas analogous to (1.6). If the expression to replace q_1 from (1.6) is denoted therein by q_2, we then obtain

$$q_2 = -\frac{\partial x}{\partial\beta}\frac{1}{h^6} \tag{1.8}$$

Let us consider the problem of stress concentration at a hole whose outline is given by the equation $\beta = 0$ by considering the state of stress unperturbed by the hole to be arbitrary and bounded at infinity. To solve the problem it is necessary to construct a function satisfying (1.3), and such that the conditions

$$\sigma_\beta = \tau_{\beta\alpha} = \mu_\beta = 0 \quad \text{for} \quad \beta = 0 \tag{1.9}$$

would be satisfied, while the stresses would tend to the unperturbed stresses far from the outline. The problem posed degenerates into the classical problem when $l = 0$. Hence, (1.3) goes over into the equation

$$\nabla^2\nabla^2 F_0 = 0 \tag{1.10}$$

and two boundary conditions must be satisfied

$$\sigma_\beta^{(0)} = \tau_{\beta\alpha}^{(0)} = 0 \quad \text{for} \quad \beta = 0 \tag{1.11}$$

where $\sigma_\beta^{(0)}$ and $\tau_{\beta\alpha}^{(0)}$ are values of σ_β and $\tau_{\beta\alpha}$ at $l = 0$. Therefore, one condition drops out in passing to the classical problem. Let us write (1.3) by expanding the coefficients of the derivatives in a Taylor series in β in the neighborhood of the outline $\beta = 0$

$$\nabla^2\nabla^2 F - l^2\nabla^2\nabla^2\nabla^2 F = \frac{1}{h_0^4}\frac{\partial^4 F}{\partial\beta^4} - l^2\frac{1}{h_0^6}\frac{\partial^6 F}{\partial\beta^6} + \ldots = 0$$

Here, only terms needed later have been written down. Let us consider the equation

$$\frac{1}{h_0^4}\lambda^4 - l^2\frac{1}{h_0^6}\lambda^6 = 0$$

The nonzero roots of this equation are

$$\lambda_{1,2} = \pm\left|\frac{h_0}{l}\right|$$

This latter equation has one root with negative real part, i.e. so that the boundary conditions will drop out in the transformation to the classical problem. Therefore, Eq. (1.3) and conditions (1.9) degenerate into (1.10) and conditions (1.11), which are called regular in [1], and the results of [1] can then be used for a sufficiently smooth outline $\beta = 0$. Let us utilize the representation A. Let F_1 determine the solution of the problem of the concentration by classical theory, i.e. by an integral of (1.10) with the boundary conditions (1.11). As the approximate solution F_2 of (1.3) let us take a function of so-called boundary layer-type in the neighborhood of the outline $\beta = 0$, namely

$$F_2 = k_1 \exp\left(-\left|\frac{h_0}{l}\right|\beta\right)$$

where $k_1 = k_1(\alpha)$. It follows from [1] that

$$\nabla^2\nabla^2 F_2 - l^2\nabla^2\nabla^2\nabla^2 F_2 = lO(1) \qquad (1.12)$$

It is here assumed that $k_1(\alpha)$ is a sufficiently smooth function to be determined. In cases similar to (1.12) we shall say that the considered condition of the problem (Eq. (1.3) in this case) is satisfied to the accuracy of quantities on the order of l or, that the error in satisfying the conditions is on the order of l. Such an error will evidently occur in complying with (1.3) if we take as its solution

$$F = F_1 + F_2$$

Let us find a function k_1 so that F would satisfy the boundary condition

$$\mu_\beta = -\frac{4^*(1-\nu)Gl^2}{1-2\nu} q_{10} \frac{\partial^4}{\partial\beta^4}\left[F_1 + k_1 \exp\left(-\left|\frac{h_0}{l}\right|\beta\right)\right] + \ldots = 0 \qquad \text{for} \quad \beta = 0$$

Here q_{10} is the value of q_1 at $\beta = 0$. It is seen from (1.7) that q_{10} vanishes at those points of the outline at which the tangent to the outline of the hole is perpendicular to the x-axis. Let us eliminate these points from consideration. The function F_1 is independent of l and will be sufficiently smooth by virtue of the condition of smoothness of the outline ; hence, the quantity $\partial^4 F/\partial\beta^4|_{\beta=0}$ is bounded ; therefore

$$k_1 = tl^4 \qquad (1.13)$$

where $t = t(\alpha)$ is a bounded quantity.

The function F_1 satisfies conditions (1.11) exactly, and F satisfies the first two conditions of (1.9) inaccurately. Let us estimate the error thus obtained. It is clear that the error from F_1 is on the order of l^2. Taking account of (1.13), we see directly that the error from F_2 is on the order of l. Therefore, F satisfies the mentioned conditions to the accuracy of a quantity on the order of l. The solution

$$\Phi = \Phi_1 + \Phi_2 \qquad (1.14)$$

for the representation B is constructed perfectly analogously. Here Φ_1 is the classical solution of the problem of concentrations, Φ_2 is a function of boundary layer-type. The solution (1.14) satisfies the system (1.1) to the accuracy of a quantity on the order of l, except, possibly, at points at which the tangent to the outline is perpendicular to the y-axis, as follows from (1.8).

Let us form a column out of the left sides of (1.6), and denote it by σ. The columns of the operators on the right sides of (1.6) are denoted by L_0 for l^0 and by L_2 for l^2. We designate the corresponding columns for the representation B by M_0 and M_2.

Let us divide the outline $\beta = 0$ into m pieces such that either $q_1 \neq 0$ or $q_2 \neq 0$ on each. Let α vary between 0 and 2π on the outline, and let α_i be the points of division, where $i = 0, 1, \ldots, m$. Evidently, there is an ε_i for each α_i such that the quantities q_1 and

q_2 would not vanish on the segment $[\alpha_i - \varepsilon_i, \alpha_i + \varepsilon_i]$. For definiteness, let us consider $q_1 \neq 0$ on those segments $[\alpha_i, \alpha_{i+1}]$ where i is even, and $q_2 \neq 0$ in the rest. Let us introduce the sufficiently smooth functions $\varphi_1(x)$ and $\varphi_2(x)$, which are mutually symmetric in the xy plane relative to the line $y = 1/2$, where $\varphi_1 \equiv 1$ in those segments $[\alpha_i + - \varepsilon_i, \alpha_{i+1} - \varepsilon_{i+1}]$ where i is even, and $\varphi_1 \equiv 0$ in those segments $[\alpha_i + \varepsilon_i, \alpha_{i+1} - \varepsilon_{i+1}]$ where i is odd.

It follows from the uniqueness theorem in classical elasticity theory that

$$L_0 F_1 = M_0 \Phi_1 \tag{1.15}$$

Formulas (1.4) and analogous formulas for the representation B, as well as the considered type of state of stress unperturbed by the hole permit the deduction that the quantities $L_2 F_1$ and $M_2 \Phi_1$ tend to zero with distance from the hole. Consequently, F_2 and Φ_2 are boundary layer-type functions, the quantities

$$L_0 F_2, \quad L_2 F_2, \quad M_0 \Phi_2, \quad M_2 \Phi_2$$

also diminish to zero far from the outline $\beta = 0$. It follows from the above that the solution of the posed problem will be the following to the accuracy of quantities on the order of l:

$$\sigma = L_0 F_1 + \varphi_1(\alpha) \; [l^2 L_2 F_1 + (L_0 + l^2 L_2) F_2] + \varphi_2(\alpha) \; [l^2 M_2 \Phi_1 + (M_0 + l^2 M_2) \Phi_2]$$

Indeed, the order of the error does not change by comparison with the solutions F and Φ, and the state of stress tends to the corresponding stress state unperturbed by the hole with distance from the hole. The problem is solved. Let us note that it is necessary to operate on two representations, because each separately does not possess sufficient generality at points where q_{10} and q_{20} (the value of q_2 at $\beta = 0$) vanish. From the mode of the solution found we can deduce that the effect of bending stresses in the problem of stress concentration at an arbitrary curved hole (with the constraints stipulated above) is on the order of l everywhere including points of the outline.

2. Let us consider an upper half-plane $y \geqslant 0$ subjected to a loading varying as $-\sin nx$ applied orthogonal to the plane at the $y = 0$ boundary. For convenience, let us introduce the quantity N by means of $N = ln$. Let us utilize the representation A. The solution by classical theory of the problem is

$$F_0 = (e_1 + e_2 y) e^{-ny} \cos nx \tag{2.1}$$

Finding the constants e_1 and e_2 from the boundary conditions we obtain

$$\sigma_x^{(0)} \big|_{y=0} = -\sin nx$$

Here $\sigma_x^{(0)}$ is the value of σ_x from (1.4) at $l = 0$. The couple-stress theory solution is

$$F = [(f_1 + f_2 y) e^{-ny} + f_3 \exp(-n[1 + N^{-2}]^{1/2} y)] \cos nx \tag{2.2}$$

Finding the constants f_1, f_2, f_3 from the boundary conditions

$$\sigma_y = -\sin nx, \quad \tau_{yx} = 0, \quad \mu_y = 0 \;\; \text{for} \;\; y = 0$$

we obtain

$$\sigma_x \big|_{y=0} = \left[-1 + \frac{4(1-\nu)([1 + N^{-2}]^{1/2} - 1)}{2(1-\nu)([1 + N^{-2}]^{1/2} - 1) + 1/2 N^{-2} [1 + N^{-2}]^{1/2}}\right] \sin nx$$

It has been shown in [6] that the Poisson coefficient ν varies between 0 and $1/2$ in couple-stress elasticity theory. Let us use the notation

$$\sigma_x^{(0)} \big|_{y=0} = \sigma_{x0}^{(0)}, \quad \sigma_x \big|_{y=0} = \sigma_{x0}, \quad \delta = \left| \frac{\sigma_{x0}^{(0)} - \sigma_{x0}}{\sigma_{x0}^{(0)}} \right|$$

and let us evaluate δ for several values of N between $^1/_{10}$ and 10 for $\nu = 0$ and $\nu = {^1/_2}$.

for $\nu = 0$

$N = {^1/_{10}}$	$^1/_8$	$^1/_6$	$^1/_4$	$^1/_2$	1	2	4	6	8	10
$\delta = 0.07$	0.105	0.18	0.32	0.74	1.07	1.26	1.3	1.36	1.33	1.33

for $\nu = {^1/_2}$

$N = {^1/_{10}}$	$^1/_8$	$^1/_6$	$^1/_4$	$^1/_2$	1	2	4	6	8	10
$\delta = 0.035$	0.053	0.09	0.17	0.4	0.73	0,92	0.98	1.05	1	1

It is seen from the tables that the correction to classical theory is essential in this problem if the wavelength of the sinusoidal loading is comparable to the characteristic length of the material l, i.e. if n is large. It is seen from (2.1) and (2.2) that in this case both the classical theory and the couple-stress theory solutions damp out rapidly with distance from the boundary. The correction in the simplest problem solved here turns out to be essential, which is not accidental. It has been shown in [7] that for elliptic equations with rapidly oscillating boundary conditions given on a sufficiently smooth boundary, the solution damps out rapidly with distance from the boundary, and can easily be constructed to any degree of accuracy. It is hence easy to observe that if the boundary conditions of the problem of oscillations are given on a sufficiently smooth boundary in couple-stress elasticity theory, then the solution damps out rapidly with distance from the boundary as the oscillations increase, and the correction to classical theory hence grows and can turn out to be essential.

The author is grateful to A. L. Gol'denveizer for supervising the research, and to Iu. N. Rabotnov for his attention.

BIBLIOGRAPHY

1. Vishik, M. I. and Liusternik, L. A., Regular degeneration and boundary layer for a linear differential equation with a small parameter. Usp. Matem. Nauk Vol. 12, №5, 1957.
2. Wilmanski, K., Asymptotic method in the theory of a disk with microstructure. Rozpr. Inz., Vol. 15, №2, 1967.
3. Mindlin, R. D., Influence of couple stresses on stress concentration. Sb. Mekhanika, №4, 1964.
4. Gol'denveizer, A. L., Theory of Elastic Thin Shells. Gostekhteoretizdat, 1953.
5. Mindlin, R. D. and Tiersten, G. F., Effects of couple stresses in linear elasticity theory. Sb. Mekhanika, №4, 1964.
6. Koiter, W. T., Couple stresses in elasticity theory. Sb. Mekhanika, №3, 1965.
7. Vishik, M. I. and Liusternik, L. A., Asymptotic behavior of the solution of linear differential equations with large, or rapidly varying, coefficients and boundary conditions. Usp. Matem. Nauk, Vol. 15, №4, 1960.

Translated by M. D. F.